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Vortex length, vortex energy and fractal dimension of superfluid turbulence at very low temperature

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Abstract

By assuming a self-similar structure for Kelvin waves along vortex loops with successive smaller scale features, we model the fractal dimension of a superfluid vortex tangle in the zero temperature limit. Our model assumes that at each step the total energy of the vortices is conserved, but the total length can change. We obtain a relation between the fractal dimension and the exponent describing how the vortex energy per unit length changes with the length scale. This relation does not depend on the specific model, and shows that if smaller length scales make a decreasing relative contribution to the energy per unit length of vortex lines, the fractal dimension will be higher than unity. Finally, for the sake of more concrete illustration, we relate the fractal dimension of the tangle to the scaling exponents of amplitude and wavelength of a cascade of Kelvin waves.

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1. Introduction

Turbulence in helium II, or superfluid turbulence, consists of a tangle of quantized vortex lines [1, 2]. Until recently, in most experiments superfluid turbulence was created in superfluid helium at rest in the presence of a heat flux, the so-called counterflow [3, 4], an interesting problem of non-equilibrium physics [5, 6]. More recently, superfluid turbulence was generated by agitating the liquid helium using grids or propellers [7–9]. Particularly interesting is the case in which the temperature T is small enough ($T < 1\text{K}$) that the normal fluid fraction of helium II is negligible; hence, viscous dissipation and mutual friction play no role. In this low temperature limit, superfluid turbulence takes its purest form: a tangle of reconnecting

vortex filaments which move under the velocity field of each other. The importance of vortex reconnections, first recognized by Schwarz [10] and later proved by Koplik and Levine [11], cannot be underestimated [12, 13]. Vortex reconnections randomize the vortex tangle and initiate the physical mechanisms of the decay of the tangle's kinetic energy in the absence of viscous losses. The first mechanism is the direct conversion of energy into sound in the form of rarefaction pulses at reconnecting events, as predicted by the condensate nonlinear Schroedinger equation model [14]. The second mechanism is a cascade of Kelvin waves of shorter and shorter wavelengths [15–23] triggered by vortex reconnections. This process of generation of small scales can proceed without significant kinetic energy losses to spatial scales which are small enough that the kinetic energy of the highly curved and cusped fragments of the vortices is radiated away as sound [24–27] (phonon emission); that is to say, ultimately kinetic energy becomes heat. In this regime, one expects the vortex tangle to exhibit fractal features, if the mentioned processes act in a self-similar way on several orders of spatial lengths. The idea that at very low temperatures the energy can be released by vortex reconnections, from smaller and smaller structures, and hence it increases the twisting and the winding of the superfluid turbulence, was originally suggested a long time ago by Feynman in his pioneering article on the applications of quantum mechanics to liquid helium [28], before we knew about fractals or the Kelvin wave cascade, and was explored in detail by Svistunov [15].

Here we propose simple geometrical models of the fractal dimension of superfluid turbulence, which represent reconnections and interactions between vortex loops and the subsequent formation at the next generation of new vortex loops and Kelvin waves on them. The models are too simple to be dynamically realistic, but sufficiently appealing for a qualitative understanding of some physical features influencing the fractal dimension. We stress that we are not attempting to develop a theory of the Kelvin wave cascade based on actual vortex dynamics, but we shall move with simpler considerations. Our motivations are the growing interest in superfluid turbulence at very low temperatures [29, 30], and previous remarks on the fractal nature of superfluid turbulence. In particular we recall the work of Kivotides *et al* [31], who numerically determined a fractal dimension larger than unity (but at finite temperature, not in the limit of absolute zero which we consider here), of Nemirovskii *et al* [32] (who considered the influence of the possible fractal dimension of the tangle on the energy spectrum of the turbulent velocity field) and of Jou *et al* [33] (who proposed an heuristic form of Vinen's generalized equation for the dynamics of a fractal vortex tangle).

Our aim is to model the fractal dimension of the tangle under the condition of constant energy, but separating the scaling behaviour of vortex length and of vortex energy during the transfer of line length to smaller and smaller length scales. The underlying physical idea is that the energetic contributions of very close parts of the vortex lines may interfere with each other, thus leading to a non-additive global result for the total energy of the loop.

First of all, we derive a general relation between the fractal dimension of the hierarchy of self-similar vortex loops and the behaviour of the vortex energy per unit length at different length scales. Afterwards, in order to be more concrete and explicit, we propose two simple models of hierarchies of self-similar loops, whose behaviour mimics in a simplified way the features of a cascade of Kelvin waves.

Our simple models are partially inspired to the well-known β -model for classical intermittent turbulence [34–37] and include the influence of geometrical and energetic aspects on the fractal dimension. We are not aware of applications of the β -model of classical turbulence to quantum turbulence. We think that this model can be useful to grasp some qualitative transfer amongst different length scales. Our approach differs from that of Svistunov [15] in that it takes a less detailed and less quantitative form, but it allows a simpler and more intuitive view of the complicated process in question.

2. Fractal dimension and behaviour of the vortex energy per unit length at different scales

Our aim is to look for an expression of the fractal dimension D_F of the vortex tangle in terms of the microscopic properties previously mentioned, namely vortex length distribution, amplitude and wavelengths of Kelvin waves, and energy density per unit length. To obtain the fractal dimension D_F of the vortex tangle, we use the standard definition [35, 38]

$$D_F = - \lim_{n \rightarrow \infty} \frac{\log(N_n/N_0)}{\log(l_n/l_0)}, \quad (2.1)$$

where N_n is the number of steps along a curve (or the number of objects of a given size) and l_n is the length of a single step (or the size of a given object).

We assume that the tangle can be statistically described as a self-similar hierarchy of loops, whose forms will be discussed in section 3. The generation with $n = 1$ corresponds to the level of the biggest vortices, which become more abundant and smaller for increasing n . We call N_n the number of vortices at the n th generation, l_n the size of each loop and E'_n the energy of each loop.

Although the specific expression for the fractal dimension depends on the details of the model, we express the fractal dimension in terms of the energy per unit length at several scales. Note that in our analysis D_F is a property of the ensemble of self-similar loops, not of a single loop. In fact, the individual loops are assumed to be regular lines, and not fractal lines. Thus, our fractal dimension characterizes the self-similarity properties of the tangle and not of individual vortex lines.

Before proposing an explicit model of hierarchies of vortex loops, we relate the fractal dimension defined geometrically in (2.1) with the variation of the energy per unit length at different length scales.

We assume that $E'_n \propto l_n^{\alpha'}$, where α' is a constant scaling exponent; this means that the energy per unit length is

$$E'_n/l_n \propto l_n^{\alpha'-1}. \quad (2.2)$$

Therefore, if $\alpha' > 1$ the contribution to the energy per unit length decreases for lower length scales (shorter l_n); the opposite is true for $\alpha' < 1$. If $\alpha' = 1$, then the energy per unit length is the same at each length scale. In principle, the exponent α' does not depend on the fractal dimension (2.1), but it becomes related to D_F if we assume the condition of constant total energy at the different loop generations, as mentioned in the introduction.

According to the previous definitions, the total energy E_n at the n th loop generation is given by $E_n = N_n E'_n$. Then the condition that the total energy is independent of n can be expressed by

$$E_n = E_{n+1}; \quad \text{hence,} \quad N_n l_n^{\alpha'} = N_{n+1} l_{n+1}^{\alpha'}. \quad (2.3)$$

If n is large enough, equation (2.1) implies that $l_n \propto N_n^{-1/D_F}$; thus, equation (2.3) leads to

$$N_n^{1-(\alpha'/D_F)} = N_{n+1}^{1-(\alpha'/D_F)}. \quad (2.4)$$

In order that this equality is true for any n , one must have that $D_F = \alpha'$. This result shows the strong connection between the energetic features of the tangle and its geometrical structure, independently of the detailed form of the loops in the hierarchy. When the contribution to the energy per unit length of the smaller length scales is smaller than the contribution of the larger scales, then $D_F > 1$, whereas in the opposite case $D_F < 1$. The case $D_F < 1$ seems

physically unacceptable because it would imply that vortex line fragments in objects perhaps similar to Cantor’s dust, which would violate the condition that the vorticity is solenoidal. This result would be supported by the numerical simulation of Kivotides *et al* [31]. The only acceptable situation is that the larger length scales contribute more to the energy per unit length than the smaller length scales. In view of the meaning of α' , we write the fractal dimension (2.1) as

$$D_F - 1 = \lim_{n \rightarrow \infty} \frac{\log(E'_n/l_n)}{\log(l_n)} \tag{2.5}$$

which, by using $l_n \propto N_n^{-1/D_F}$, can also be written as

$$\frac{1 - D_F}{D_F} = \lim_{n \rightarrow \infty} \frac{\log(E'_n/l_n)}{\log(N_n)}. \tag{2.6}$$

In the next section, we shall introduce a model of loop generations which relates the fractal dimension to the amplitude and the wavelength of the Kelvin waves.

3. Geometric and energetic assumptions

We assume that, as the mutual interaction of vortices induces the formation of helical Kelvin waves along the vortex line which also undergo breaking and reconnection processes, the tangle can be described as an ensemble of self-similar objects. Neglecting boundaries, we assume that these objects are closed vortex loops. Svistunov [15] has also considered the same point of view, but with different transformation rules than those we consider here, as we shall discuss below.

The loops can be entangled among themselves in complex topological ways [39]. Here, we focus our attention only on geometrical properties such as vortex length, vortex number, amplitude and wave-number of Kelvin waves, and energy per unit length. We do not consider the topological details of the vortex entanglement.

We envisage that the generation of vortex loops takes place according to the following rules.

- (i) We take as reference configuration a collection of N_0 vortex loops of length $L_0 = 2\pi R_0$, where R_0 is the average curvature radius of the loop, along which an helical structure of N'_0 helical turns lies, all turns being of radius R'_0 . This structure models in a simple way the formation of Kelvin helical waves along vortices, where R'_0 is the amplitude and $h_0 = L_0/N'_0$ is the wavelength of waves.
- (ii) The next generation is assumed to be composed of $N_0 r$ smaller vortex loops (where r is a multiplication parameter) of length L_0/β (where β is another parameter). The following generation consists of $N_0 r^2$ loops of lengths L_0/β^2 , and so on. Thus, the n th generation is composed of $N_n = r^n N_0$ vortex loops of lengths $L_n = L_0/\beta^n$ (see figure 1). Note that the generation of smaller vortex loops at the n th generation comes from the interactions and reconnections among vortex loops of $(n - 1)$ th generation (the dots in each loop of figure 1 denote the reconnection points). The parameters r and β thus depend on the details of the dynamics.
- (iii) Besides the above set of rules which determine the number N_n of loops in the n th generation and the average radius R_n of the main circle of the loop, it is necessary to give a second set of rules for N'_n and R'_n , which are the number of helical turns and the average radius

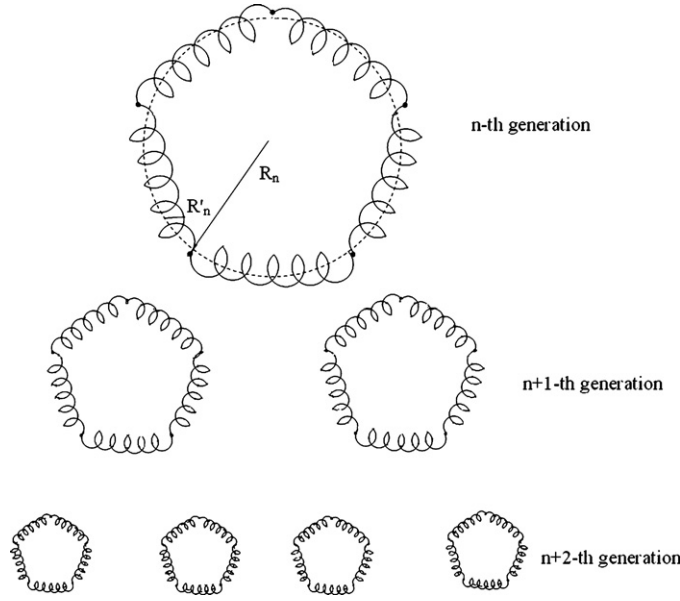


Figure 1. Each vortex loop at n th generation with the average curvature radius R_n generates $r = 2$ vortex loops at $(n + 1)$ th generation, and so on at the next generation. In the figure a constant number of helical turns $N'_n = N'_0$ ($\alpha = 0$) is assumed, while the radius R'_n decreases according to the rule $R'_n = R'_0/\beta^{\gamma n}$.

of each turn. We interpret R'_n as the amplitude of a Kelvin wave and $h_n = L_n/N'_n$ as the wavelength. We assume that each loop has $N'_n = N'_0(r')^n$ helical turns, and that the radius of the helical structure at the n th generation scales as $R'_n = R'_0/\beta^n$, i.e. it scales in a different proportion than the curvature radius of the total loop. We introduce two scaling coefficients α and γ , setting $r' = r^\alpha$ and $\beta' = \beta^\gamma$ (in figure 1, for the sake of simplicity, it is assumed that the number of helical turns in each vortex loop is constant, that is $\alpha = 0$).

- (iv) In the breaking and reconnection processes the total energy remains unchanged because at the considered length scales there is no friction, hence no energy dissipation. Thus, we impose that $E_{n+1} = E_n$ for all values of n . This assumption has a limiting length scale because at some very small length scale sound radiation becomes relevant and the fractalization process stops.

We stress that the first three rules are not supposed to model actual dynamical processes; they are only meant to explore the possible consequences of two physical processes which, in this context of superfluid turbulence at very low temperatures, are still poorly understood: a direct cascade, leading from bigger to smaller loops, and an inverse cascade, leading from smaller to bigger loops. Concerning the fourth step, another plausible choice could be the invariance of the total length, instead on the invariance of the energy; this provides another way to examine the fractal dimension which leads to $D_F = 1$. This result follows from the general arguments of section 2. The assumption that the total length of the vortex line is the same at each step n means

$$N_n l_n = N_{n+1} l_{n+1}. \tag{3.1}$$

Equation (2.1) implies that $l_n \propto N_n^{-1/D_F}$ for n high enough, so equation (3.1) becomes

$$N_n^{(1-1/D_F)} = N_{n+1}^{(1-1/D_F)}. \quad (3.2)$$

Since relation (3.2) is true for every n high enough, then $D_F = 1$. In our opinion it is an interesting result because it shows that the fractal dimension is 1 if the whole vortex line length is kept constant, that is if vortices themselves do not contribute to lengthening or shortening of the total vortex length. But, as pointed out in this paper, in the low temperature limit it seems more plausible the invariance of the total energy with respect to the invariance of the total length.

Since the total number of vortex loops at the n th generation is N_n , the total energy stored on the vortex loops of the n th generation will be

$$E_n = N_n E'_n \quad (3.3)$$

where E'_n is the energy of the loop at the n th generation. The length of a single vortex in the n th generation is

$$l_n = 2\pi N'_n R'_n \sqrt{1 + \left(\frac{R_n}{R'_n N'_n}\right)^2} = 2\pi N'_0 R'_0 \frac{r^{n\alpha}}{\beta^{n\gamma}} \sqrt{1 + \left(\frac{R_0}{N'_0 R'_0}\right)^2 \left(\frac{\beta^{\gamma-1}}{r^\alpha}\right)^{2n}}. \quad (3.4)$$

Finally, we need to specify the energy of each loop. Of course, the energy of a loop depends on the helical structure that wraps the unperturbed loop of average radius R_n , that is, on the number of helical turns N'_n or, equivalently, on the pass of helices. Unfortunately we are unable to calculate this energy analytically. The recent work of Maggioni *et al* [40] shows that, even for the simpler case of a non-fractal, single vortex filament, the numerical calculation of this energy is difficult, as it converges very slowly. For this reason, we propose three possible approximate scenarios for the energy E'_n of a loop at the n th generation.

- (i) The first scenario assumes that E'_n is proportional to the length of the circular axis of the helical loop:

$$E'_n = \frac{\rho_s \kappa^2}{4\pi} L_n \left[\log\left(\frac{8R_n}{a_0}\right) - 1.615 \right], \quad (3.5)$$

where ρ_s is the mass density of the superfluid component, κ is the quantum of vorticity ($\kappa = h/m$, with h Planck's constant and m the mass of the helium atom) and a_0 is the radius of the core of the vortices (of the order of the atomic radius) [1].

- (ii) The second scenario is to assume that for large-amplitude helical turns close to each other ($R'_n \gg h_n$) the helical loop may be considered as a solenoid, and that its internal energy E'_n is of the order of its volume, $2\pi R_n \pi R_n'^2$, times the density of the kinetic energy of the superfluid induced by the polarized coil. Using for the induced velocity an expression analogous to that for the magnetic field in a solenoid we can write for the induced velocity $v_{sl} \approx \kappa/h_n$ and obtain

$$E'_n = \pi^2 \rho_s \kappa^2 R_n \left(\frac{R'_n}{h_n}\right)^2. \quad (3.6)$$

- (iii) The third scenario is a more flexible prescription, incorporating also the features of the helical structure. The total length of the deformed circle, for n sufficiently high, is

$$l_n = 2\pi N'_n R'_n \sqrt{1 + \left(\frac{h_n}{2\pi R'_n}\right)^2}, \quad (3.7)$$

with $h_n = L_n/N'_n$ being the pass of the helices at the n th generation, which can be interpreted as the wavelength of the Kelvin wave, whose amplitude is R'_n . We assume

that the whole helical length contributes to the energy, but multiplied by a dimensionless factor depending on the length scale, as represented, for instance, by R'_n , which modulates the relative influence of the helical turns on the energy of the loop. Thus, instead of (3.5), we take

$$E'_n = \frac{\rho_s \kappa^2}{2} N'_n R'_n \left(\frac{R'_n}{R_0}\right)^\chi \sqrt{1 + \left(\frac{h_n}{2\pi R'_n}\right)^2} \left[\log\left(\frac{8R'_n}{a_0}\right) - \delta \right], \quad (3.8)$$

where δ is a constant of the order of 1.6. Here the ratio $(R'_n/R_0)^\chi$ with $\chi > 0$ ensures that the smaller the length scale is (i.e. the smaller R'_n is), the smaller is the contribution to the energy; if $\chi < 0$, smaller length scales have larger contributions to the energy, and $\chi = 0$ indicates that the energy is proportional to the loop length. Of course, more complicated models could be assumed instead of $(R'_n/R_0)^\chi$. The actual value of χ should be obtained from a first principles calculation of the energy of helical loops of different radii and with different separations between successive helical turns, but, as we said before, this is very complicated and goes beyond the simple, limited task which we set at this early stage of investigation.

Since $N_n/N_0 = r^n$ and $l_n/l_0 = l_n/2\pi R_0$, when l_0 is assumed to be $2\pi R_0$, the fractal dimension (2.1) becomes

$$D_F = - \lim_{n \rightarrow \infty} \frac{n \log r}{\log\left(\left(N'_0 R'_0/R_0\right) \left(r^{\alpha n} / \beta^{\gamma n}\right) [1 + [R_0/(R'_0 N'_0)]^2 (\beta^{\gamma-1} / r^\alpha)^{2n}]^{1/2}\right)}. \quad (3.9)$$

Note that here the limit is taken keeping in mind that, when n becomes large enough, the radius of the loop R_n cannot be smaller, or of the same order of the vortex core radius a_0 . We distinguish essentially two cases of physical interest: the long wavelength limit ($h_n \gg R_n$) and the large amplitude limit ($h_n \ll R_n$), corresponding respectively to Kelvin waves whose wavelengths are larger or smaller than their amplitudes.

Long wavelength limit ($h_n \gg R'_n$ or $\beta^{\gamma-1} > r^\alpha$)

In the limit of Kelvin waves with small amplitude and large wavelength, the physically most plausible scenarios are the first (3.5) and the third (3.8).

In the first scenario (3.5), under the condition $E_{n+1} = E_n$, leads to

$$[(n \ln \beta + a)] = \frac{r}{\beta} [(n+1) \ln \beta + a], \quad (3.10)$$

which is true for any value of n only if $r = \beta$ needs. It follows that the condition $h_n \gg R_n$, or $\beta^{\gamma-1} > r^\alpha$ can be read in terms of γ and α as $\gamma - \alpha > 1$, and the value of the fractal dimension can be obtained from (3.9):

$$D_F = 1 \quad \text{if} \quad \gamma - \alpha > 1. \quad (3.11)$$

It is reasonable that $D_F = 1$ because in this scenario the interference between neighbouring helical turns tends to vanish.

In the third scenario, multiplying (3.8) times N_n and requiring that $E_{n+1} = E_n$, i.e. $N_{n+1} E'_{n+1} = N_n E'_n$, we are lead to

$$\begin{aligned} & \frac{r^{(\alpha+1)(n+1)}}{\beta^{\gamma(\chi+1)(n+1)}} \sqrt{1 + \left(\frac{R_0}{N'_0 R'_0}\right)^2 \left(\frac{\beta^{\gamma-1}}{r^\alpha}\right)^{2(n+1)}} [-(n+1)\gamma \log \beta + a] \\ &= \frac{r^{(\alpha+1)n}}{\beta^{\gamma(\chi+1)n}} \sqrt{1 + \left(\frac{R_0}{N'_0 R'_0}\right)^2 \left(\frac{\beta^{\gamma-1}}{r^\alpha}\right)^{2n}} [-n\gamma \log \beta + a]. \end{aligned} \quad (3.12)$$

The second term under the square root in (3.12) is dominant and the relation between r and β becomes

$$r = \beta^{1+\gamma\chi}. \tag{3.13}$$

Substituting this relation into $\beta^{\gamma-1}/r^\alpha > 1$ one gets $(\gamma - \alpha - \alpha\gamma\chi - 1)/((1 + \gamma\chi)) > 0$.

Using (3.4) and $N_n/N_0 = r^n$ we have

$$D_F = 1 + \chi\gamma \geq 1 \quad \text{if} \quad \chi \geq 0. \tag{3.14}$$

Note that a negative value of χ implies $D_F < 1$, and that the value of χ cannot be less than $-1/\gamma > -1$. The result that $\gamma > 1$ comes from the relation $(\gamma - \alpha - \alpha\gamma\chi - 1)/((1 + \gamma\chi)) > 0$. The conclusion that $D_F < 1$ is not physically reasonable: it would imply that the vortex tangle becomes similar to Cantor’s dust and would violate the solenoid condition (a vortex is a closed loop or terminates on boundaries, but here we have no boundaries).

For the sake of completeness, we also consider scenario 2 (3.6) although this expression of the energy seem to be physically inadmissible in the long wavelength limit. The constraint $E_{n+1} = E_n$ leads to $r^{1+2\alpha} = \beta^{2\gamma-1}$ and $\gamma - \alpha - 1 > 0$. Substitution into (3.9) yields

$$D_F = \frac{2\gamma - 1}{2\alpha + 1} > 1, \tag{3.15}$$

(as $D_F - 1$ is positive when $\gamma > \alpha + 1$).

Large amplitude limit ($h_n \ll R'_n$ or $\beta^{\gamma-1} < r^\alpha$)

Suppose that Kelvin waves have amplitude larger than the wavelength. This means that the helical turns which wrap loops at the n th generation must have a radius R'_n larger than their pass h_n . The most plausible assumptions for the energy seem scenarios (3.6) and (3.8).

Using (3.6) and $E_{n+1} = E_n$ we obtain

$$\left[\pi \left(\frac{r'_0 N'_0}{L_0} \right)^2 \left(\frac{r^\alpha}{\beta^{\gamma-1}} \right)^{2n} \right] = \frac{r}{\beta} \left[\pi \left(\frac{r'_0 N'_0}{L_0} \right)^2 \left(\frac{r^\alpha}{\beta^{\gamma-1}} \right)^{2n+2} \right], \tag{3.16}$$

which is valid at any generation n if $r^{1+2\alpha} = \beta^{2\gamma-1}$. Substitution into (3.9) gives

$$D_F = \frac{2\gamma - 1}{\alpha + \gamma} < 1, \tag{3.17}$$

provided that $\gamma - \alpha < 1$, which, as we have already said, seems physically unpalusible.

Using (3.8) we obtain again (3.12), but here with the condition

$$\beta^{\gamma(1+\chi)} = r^{\alpha+1}. \tag{3.18}$$

Substitution into the inequality $\beta^{\gamma-1}/r^\alpha < 1$ yields $(\gamma - \alpha - \alpha\gamma\chi - 1)/(\gamma(1 + \chi)) < 0$.

Then, substituting (3.18) into (3.9), we obtain

$$D_F = \frac{1 + \chi}{1 - \alpha\chi}, \tag{3.19}$$

which implies that

$$\frac{D_F - 1}{D_F} = \frac{\chi(1 + \alpha)}{(1 + \chi)} \tag{3.20}$$

making apparent that

$$D_F \geq 1 \quad \text{if} \quad \chi \geq 0, \tag{3.21}$$

(otherwise $D_F < 1$). The last conclusion requires $-1 < \chi < 0$, because $0 < 1/D_F < (1 + \alpha)/(\gamma(1 + \chi))$.

Again, although scenario 1 seems to be unphysical, we remark for the sake of completeness that the condition $r = \beta$, obtained below equation (3.10), leads to

$$D_F = \frac{1}{\gamma - \alpha} > 1 \quad (3.22)$$

provided that $\gamma - \alpha < 1$.

4. Concluding remarks

Finally, the following two comments are worth mentioning. The first is that, at sufficiently low temperatures, the energy-conserving process that breaks or lengthens vortices does not continue indefinitely, but terminates at sufficiently small scales, where a significant amount of energy is dissipated as sound. The dependence of this energy radiation upon the length scale has been studied by Vinen [24], and Kozik and Svistunov [25–27]. According to Vinen’s analysis, sound radiation becomes relevant at length scales of the order of $l_{\min} \simeq (\kappa^3/\epsilon)^{1/4}$, where κ is the quantum of circulation and ϵ is the energy communicated to the system per unit volume and time, which is proportional to L^2 . Thus, l_{\min} is proportional to $L^{-1/2}$. Thus, sound emission limits the Kelvin wave cascade process considered here. In classical turbulence, viscous dissipation plays a similar role, and determines the smallest scale $l_{\text{diss}} \sim [\nu^3/\epsilon]^{1/4}$ for which the celebrated Kolmogorov scaling is valid [34, 35].

Second, the correlation between Kelvin waves has not been considered in this simple model; in [32], however, it has been argued that it could play a significant role in the fractal properties; it would be interesting to explore their contribution in a more detailed model. Our model is too simple to do so; it must be stressed that reconnections play a decisive role in breaking bigger loops into smaller loops; without reconnections, energy and momentum conservation would forbid this cascade [15].

In summary, we have proposed some toy models which allow us to interpret the fractal dimension of a vortex tangle in energetic terms. Their energy is not proportional to the vortex length—because of the mutual interference of very close parts of the vortex line—and energy, rather than the length, is conserved in the breaking and recombination of vortices. For example, very recent work by Maggioni *et al* [40] has demonstrated that for complex vortex structures such as vortex coils and vortex knots, the energy per unit length is not constant; a similar effect may occur on vortex filaments in superfluid turbulence. We have determined a relation between the fractal dimension and the influence of the smaller length scales on the total energy. If this influence is smaller than that of the bigger length scales, the fractal dimension is higher than 1. This result can be understood in an intuitive way, because energy restrictions do not limit the presence of many small and complicated vortex loops, which tend to fill a proportion of space higher than a simple geometrical line. In contrast, if smaller scales contribute considerably to the energy, energy restrictions limit the formation of these scales and vortex loops will be relatively large and simple. Our results show that when small length scales contribute relatively less to the energy than the long scales, the fractal dimension D_F is larger than 1. The opposite is not true; one could have a fractal dimension higher than 1, but with an essentially linear relation between energy and length. The logarithmic dependence of the fractal dimension on the behaviour of the energy per unit length at different scales may allow us to obtain a reasonably physical result for D_F without knowing in full detail the exact form of the energy contribution of loops at different scales.

Another interesting result is pointed out in section 3, where we investigate what happens if the whole vortex line length is kept invariant at each generation n , instead of the total energy. The result is that the fractal dimension has to be 1, that is the assumption that each vortex

does not contribute to the lengthening or shortening of the other vortices means that vortices are not fractals.

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